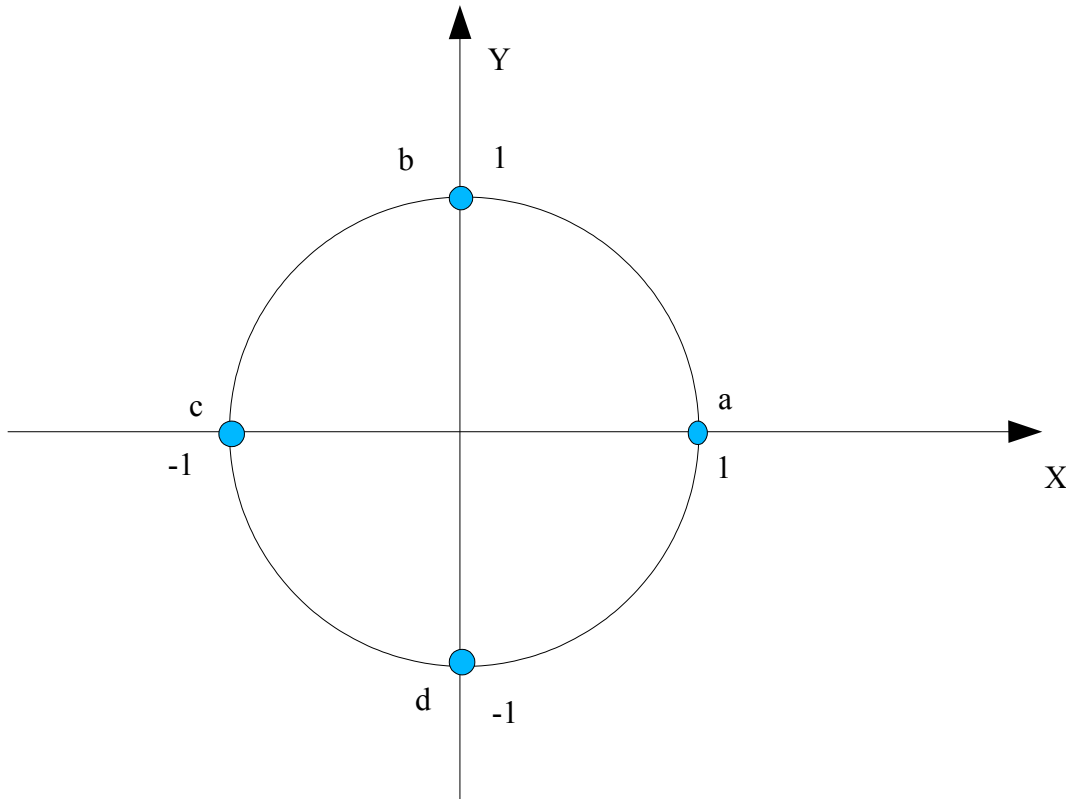


The **unit circle** is a circle whose radius is 1 unit and whose center is at the origin of the XY coordinate system. It is important since it provides a visual understanding of the relationships between the trigonometric functions. These functions are used extensively in higher mathematics and in physics in particular to model harmonic phenomena, examples include vibrating strings, sound, light, and alternating current.



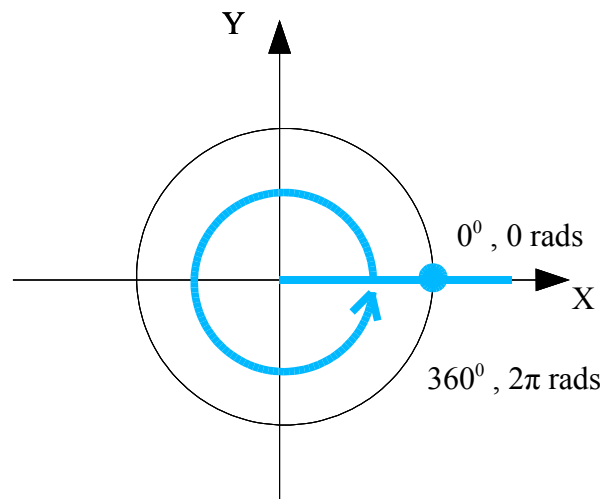
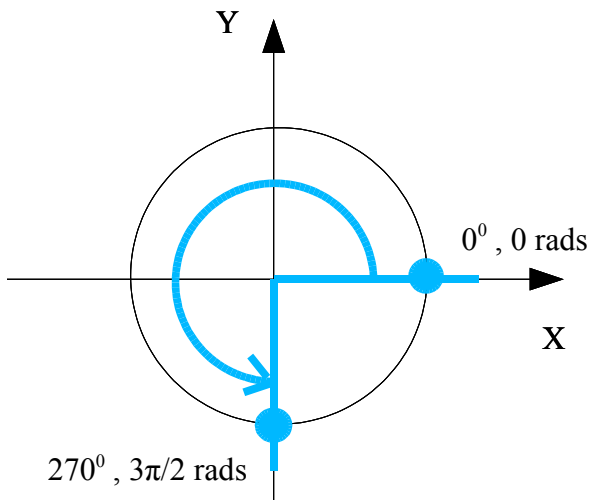
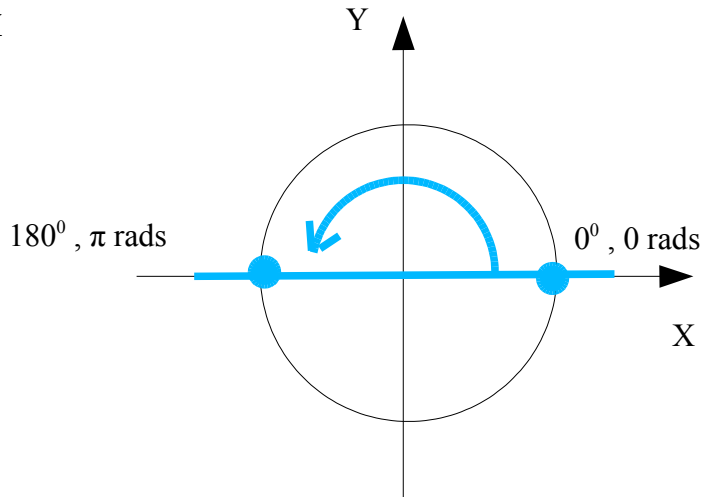
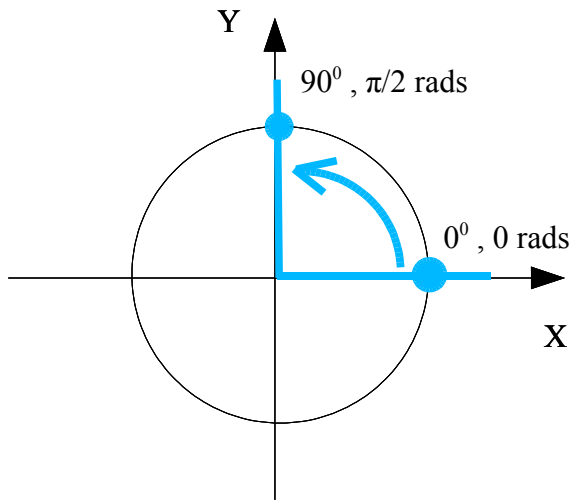
Observation 1: the unit circle intersects the X and Y axis at

- a) $X = 1 \quad Y = 0$;
- at b) $X = 0 \quad Y = 1$;
- at c) $X = -1 \quad Y = 0$;
- and at d) $X = 0 \quad Y = -1$

Observation 2: the circumference of this circle is 2π (see Circles) which is also the full angle of this circle. The points on this circle moving counter clockwise from $X = 1 \quad Y = 0$ represent the angles :
 0 rads, $\pi/2$ rads, π rads , $3\pi/2$ rads and 2π rads respectively (or 0° , 90° , 180° , 270° and 360° respectively; rads are more convenient to use, however degrees should be learned as well).

Angular measure on the unit circle:

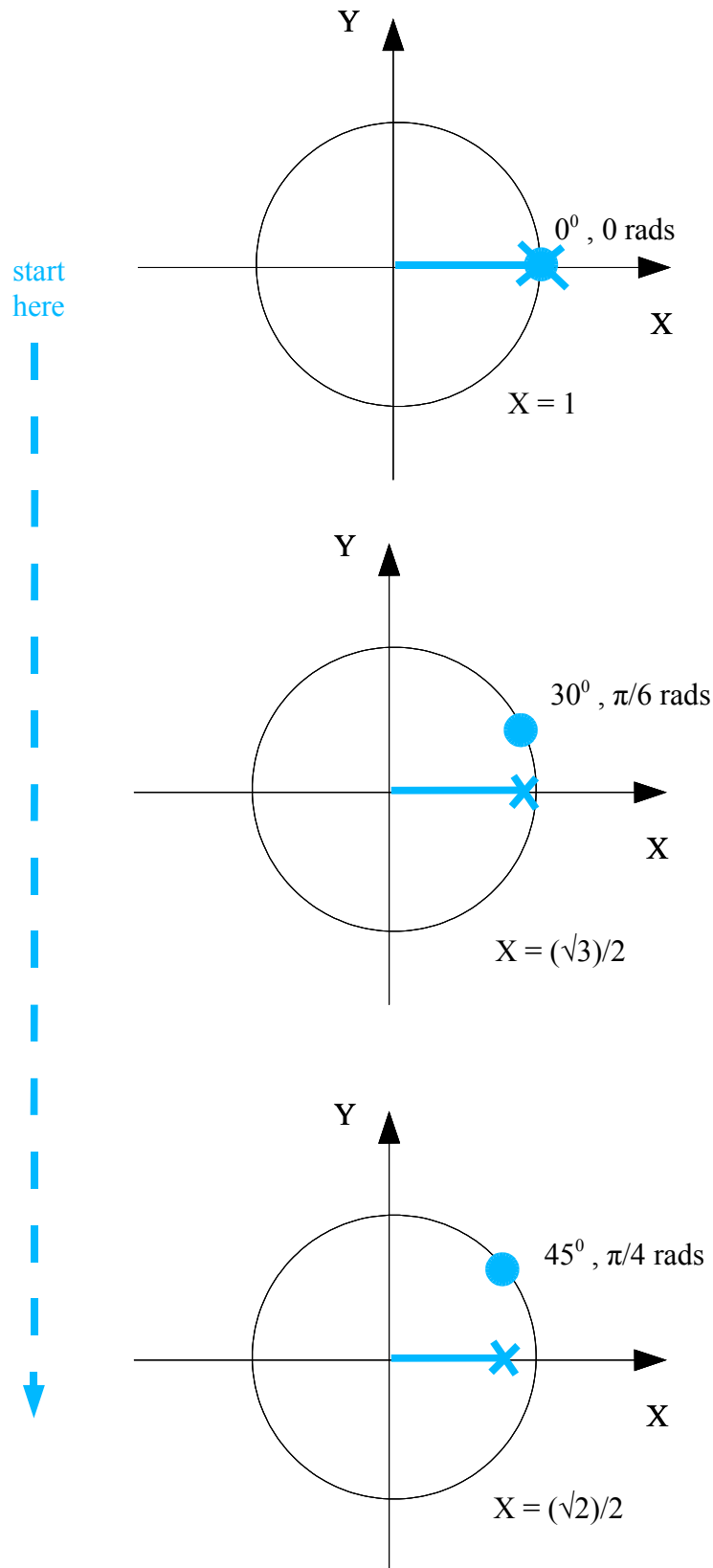
All angles start from the X axis and measured counter clockwise are positive, measured clockwise are negative.

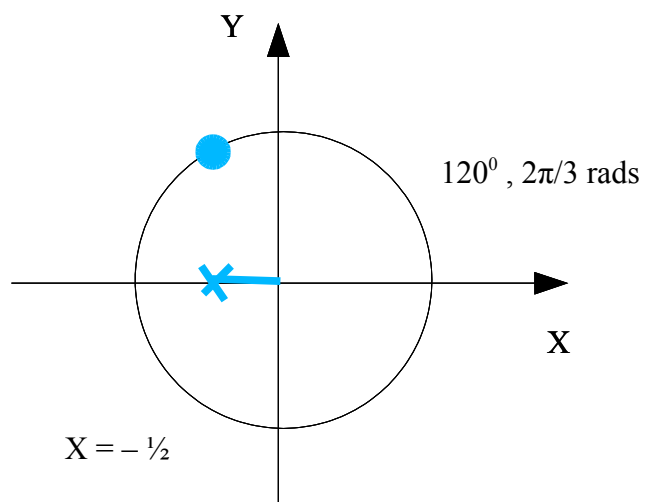
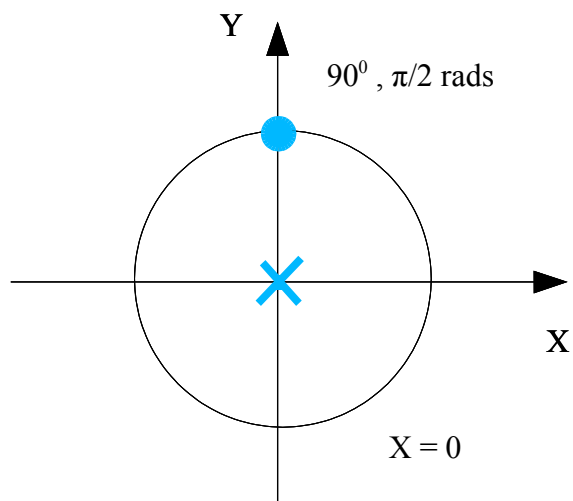
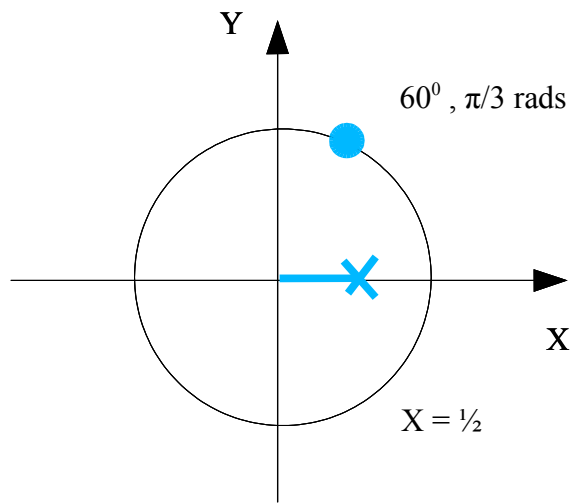


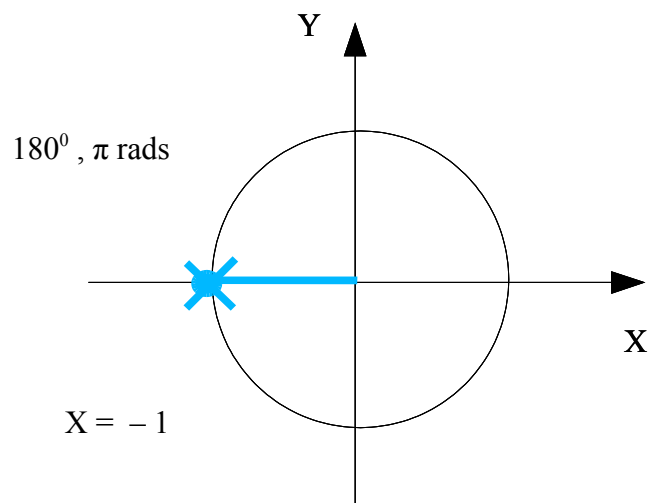
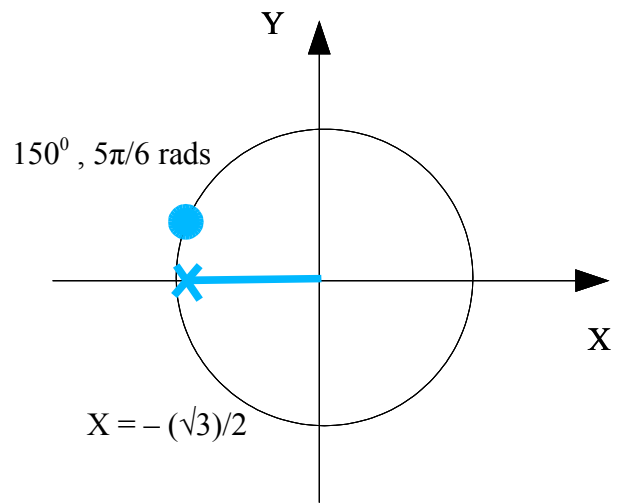
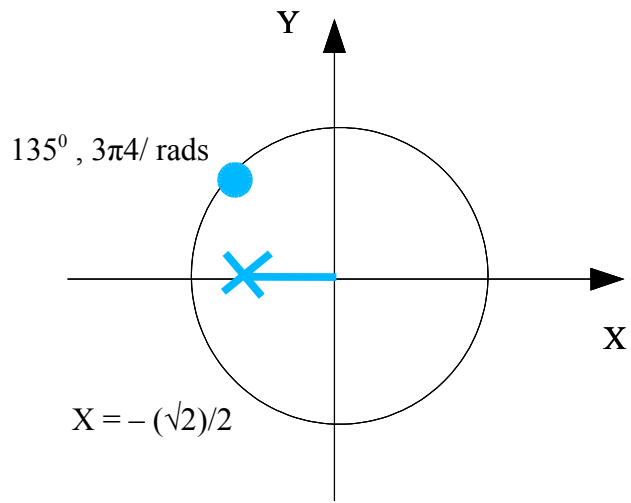
The angle 90° at $(x,y) = (0,1)$ can be represented using $450^{\circ}, 540^{\circ}$, etc., or $5\pi/2, 9\pi/2$, etc., with as many revolutions about the circle as we wish. The important point here is the same point on the circle can be measured by an endless number of angles.

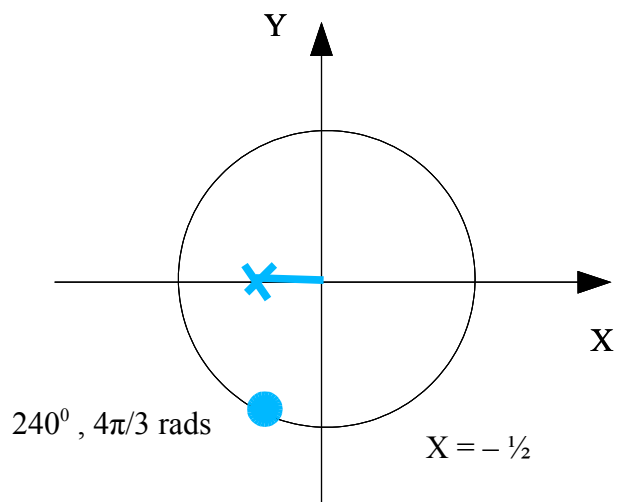
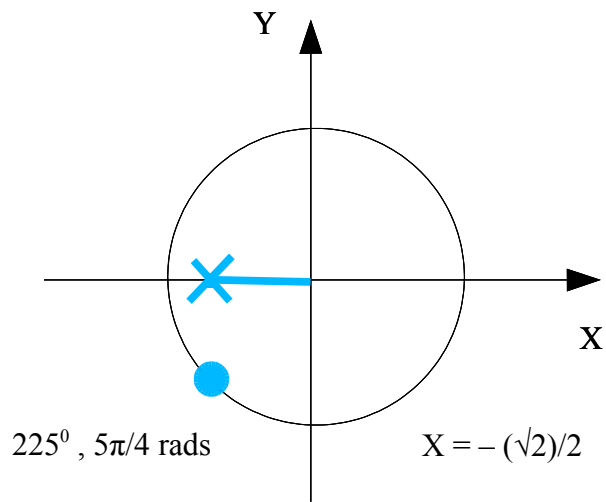
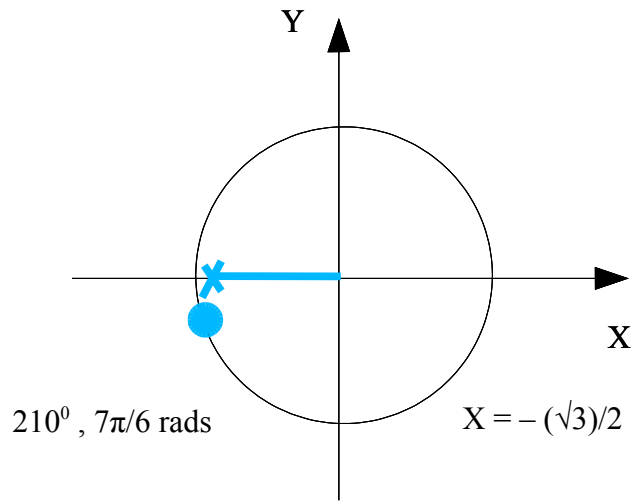
Given any angle on the unit circle, adding or subtracting multiples of 360° , or multiples of 2π , describes the same (x,y) point on the circle.

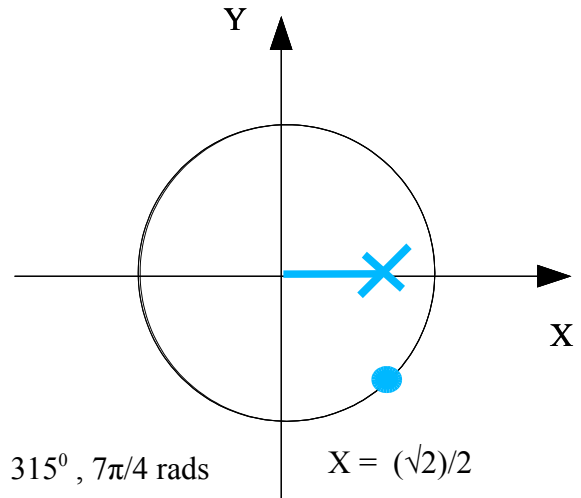
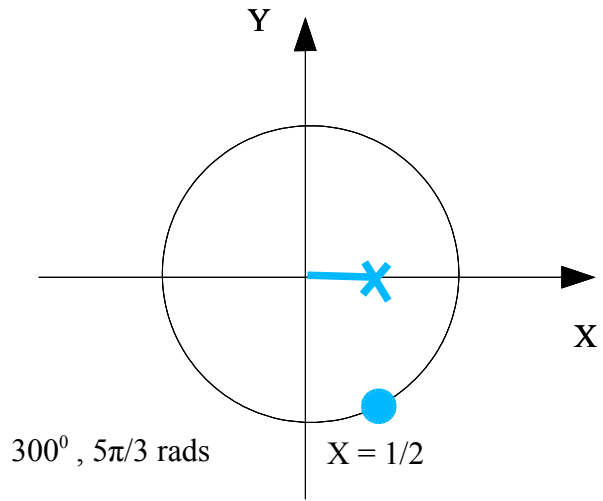
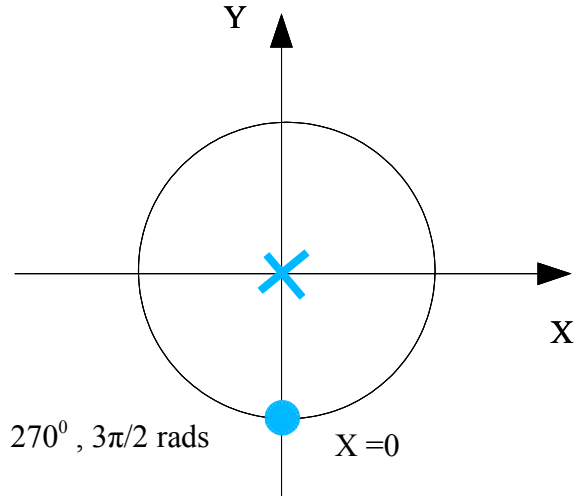
Watch the value of the x coordinate of the point on the unit circle as we measure angles around the unit circle: note how the length of the blue line along the X axis changes as we move around the circle.

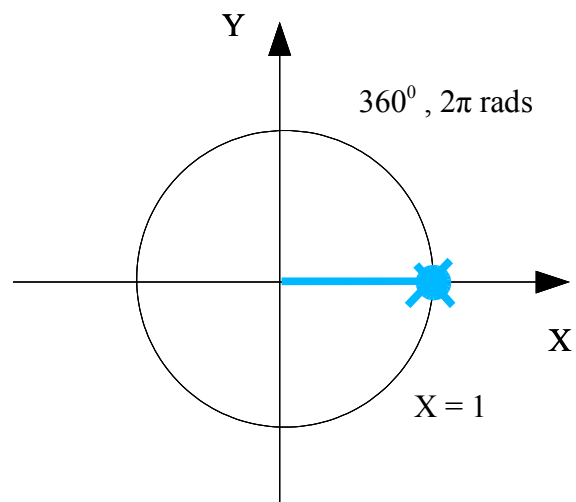
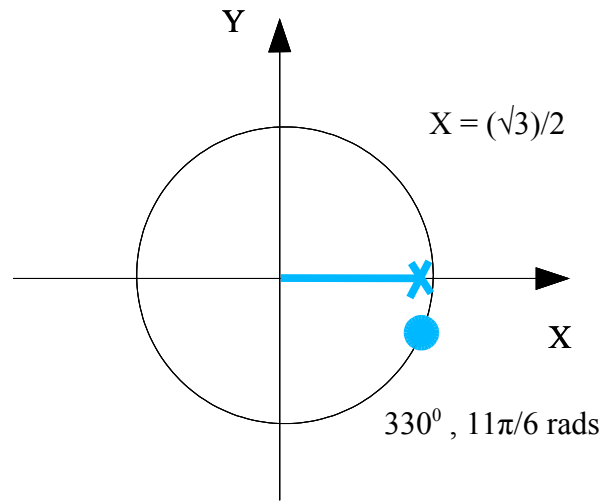






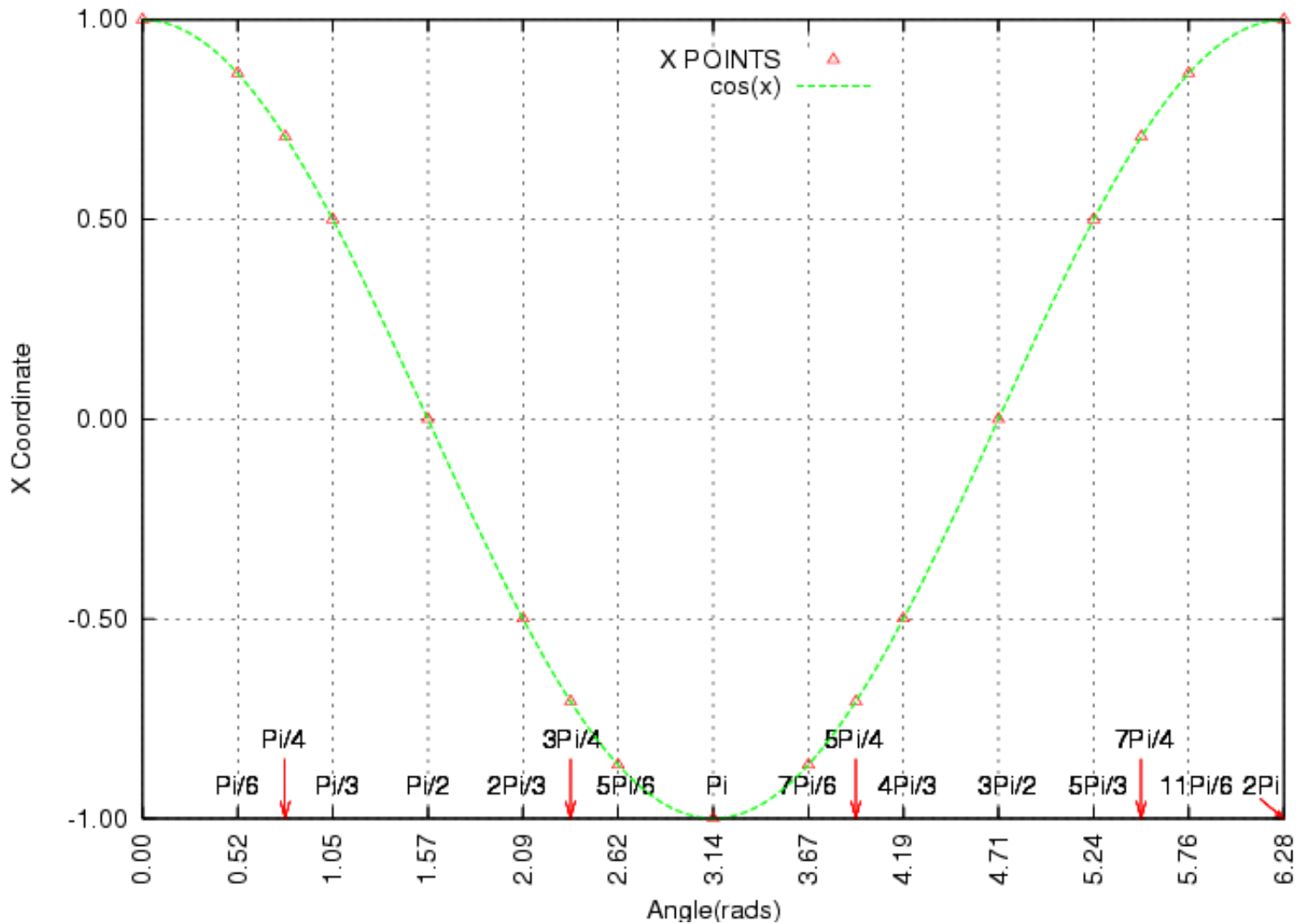




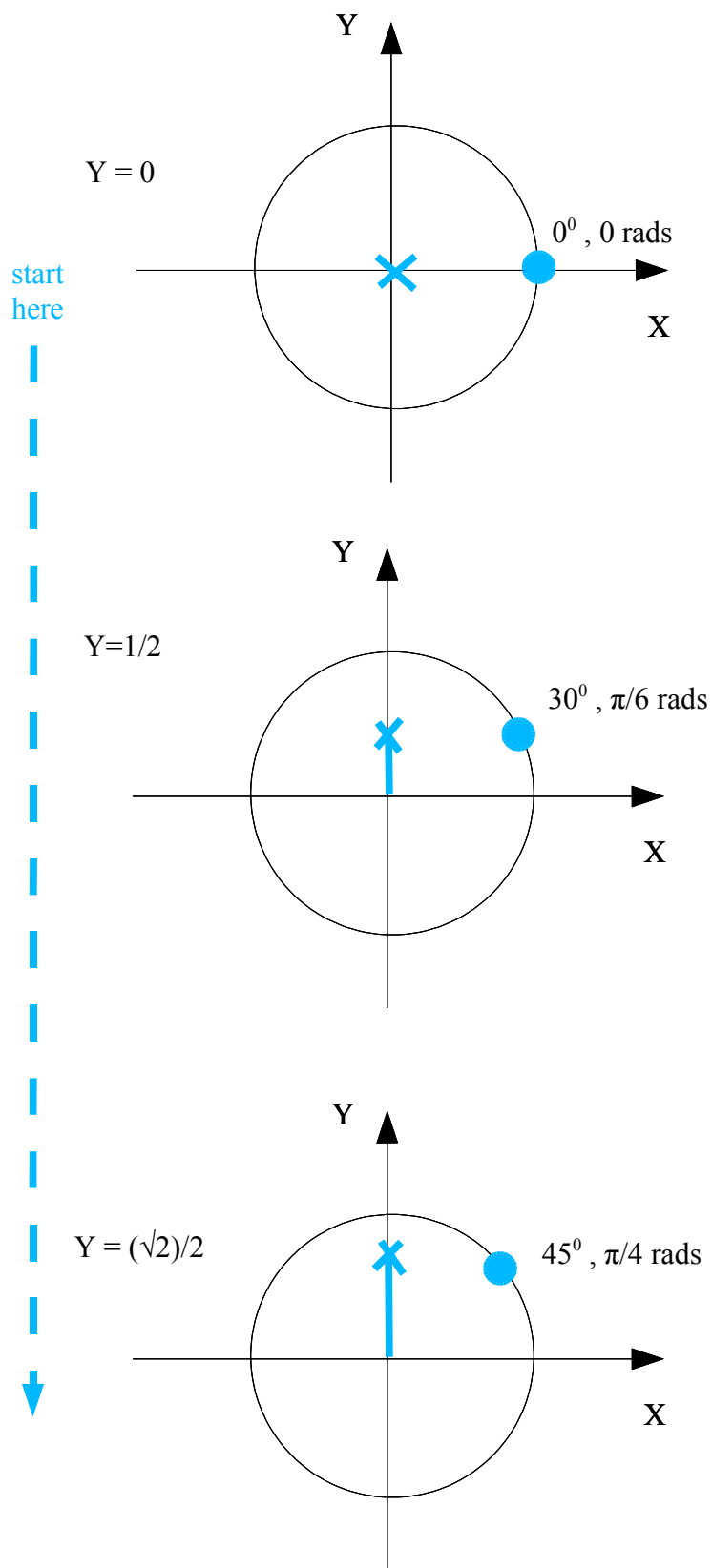


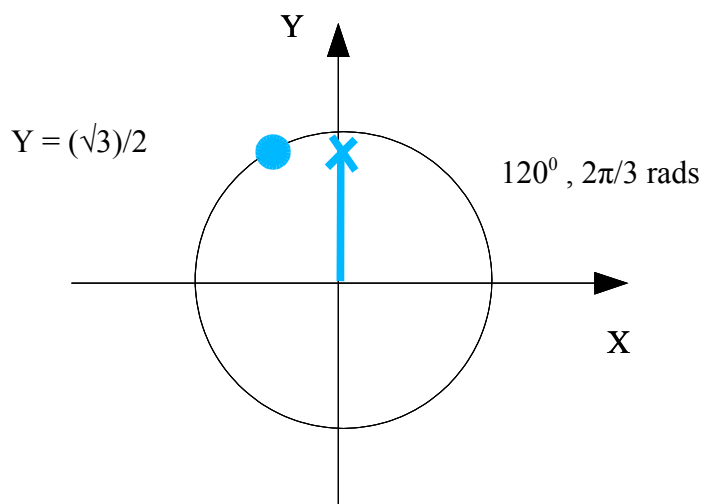
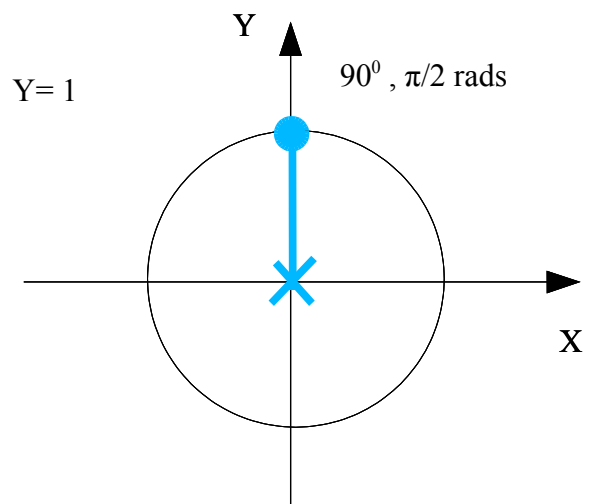
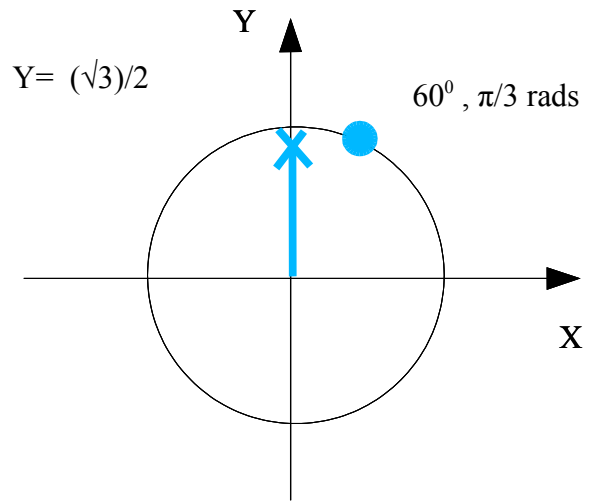
We've completed one revolution, the next page gives a table of these X values followed by a graph of these X values versus the angle measured.

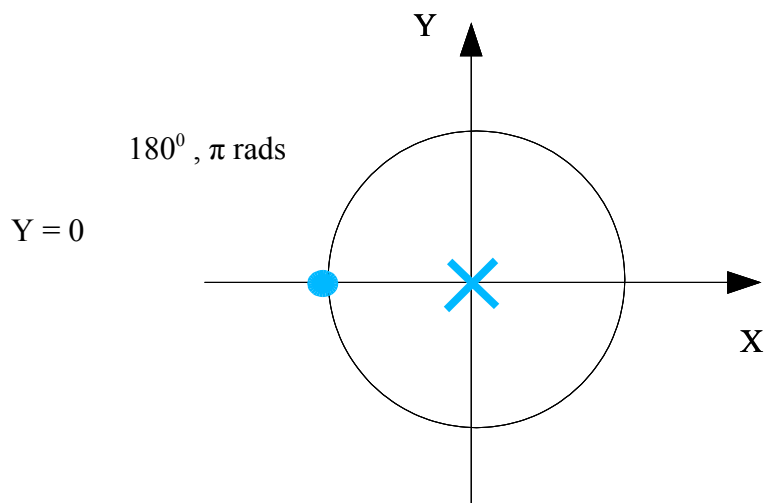
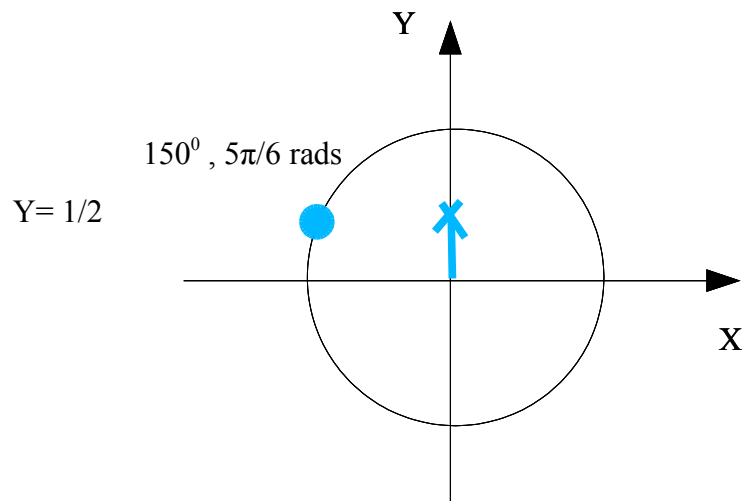
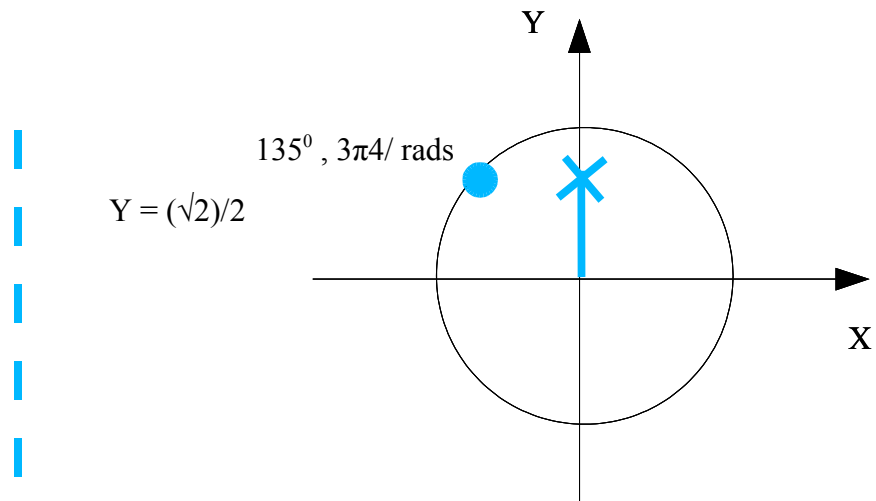
<u>Angle degrees</u>	<u>Angle rads</u>	<u>X</u>
0	0	1
30	$\pi/6$	$(\sqrt{3})/2$
45	$\pi/4$	$(\sqrt{2})/2$
60	$\pi/3$	$1/2$
90	$\pi/2$	0
120	$2\pi/3$	$-1/2$
135	$3\pi/4$	$-(\sqrt{2})/2$
150	$5\pi/6$	$-(\sqrt{3})/2$
180	π	-1
210	$7\pi/6$	$-(\sqrt{3})/2$
225	$5\pi/4$	$-(\sqrt{2})/2$
240	$4\pi/3$	$-1/2$
270	$3\pi/2$	0
300	$5\pi/3$	$1/2$
315	$7\pi/4$	$(\sqrt{2})/2$
330	$11\pi/6$	$(\sqrt{3})/2$
360	2π	1

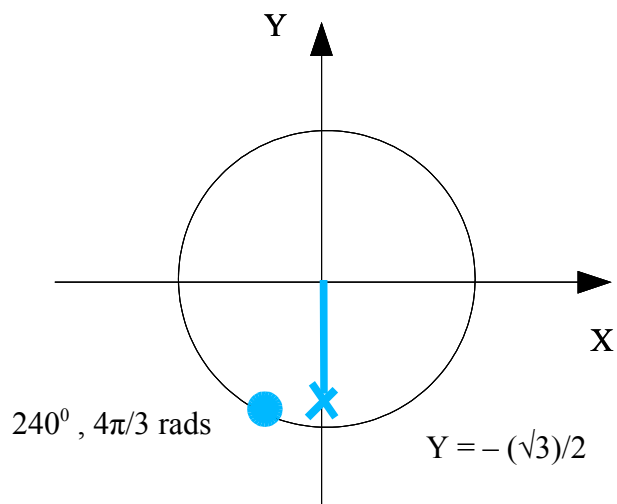
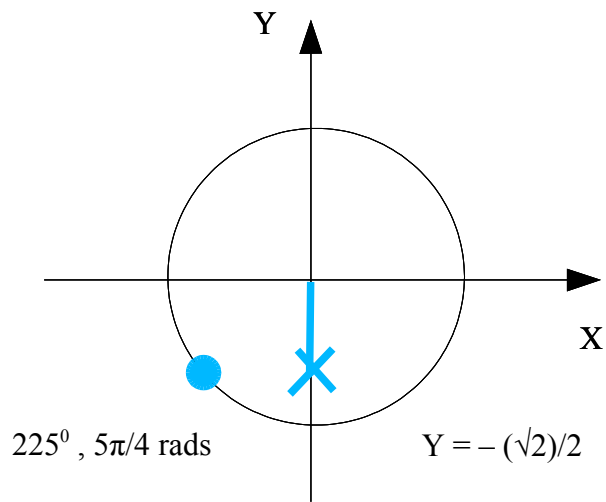
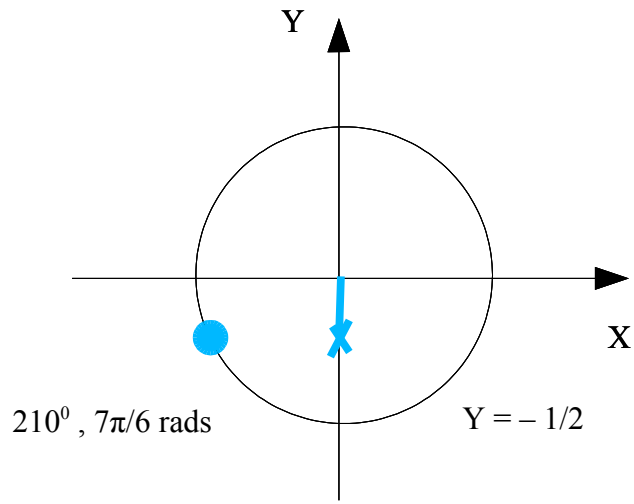


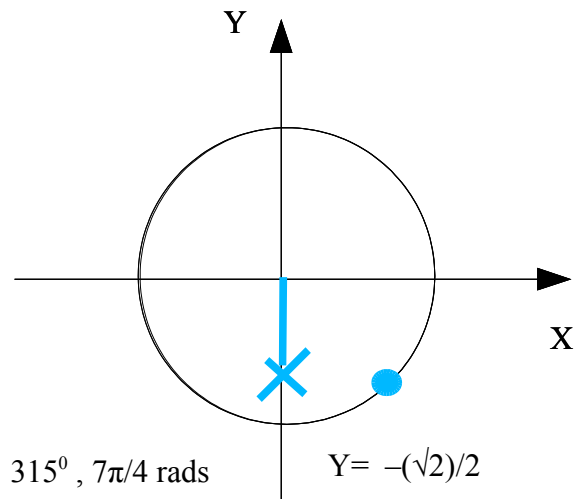
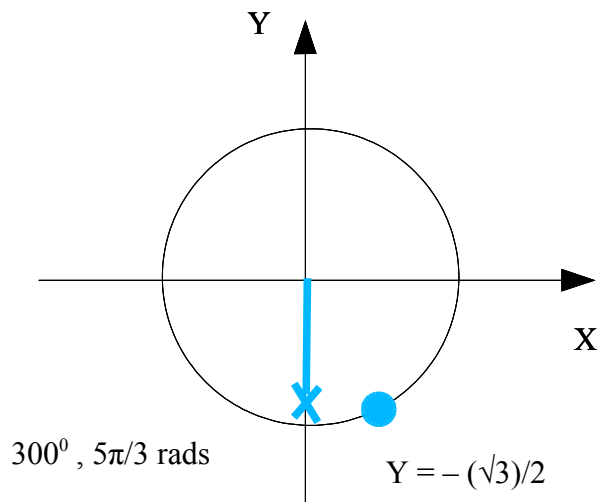
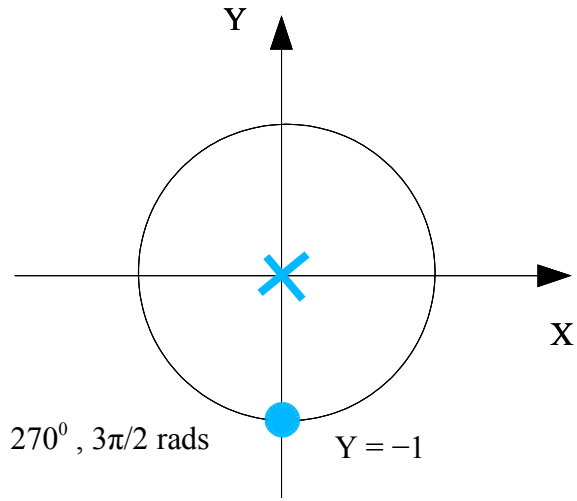
Watch the value of the y coordinate of the point on the unit circle as we measure angles around the unit circle: note how the length of the blue line along the Yaxis changes as we move around the circle.

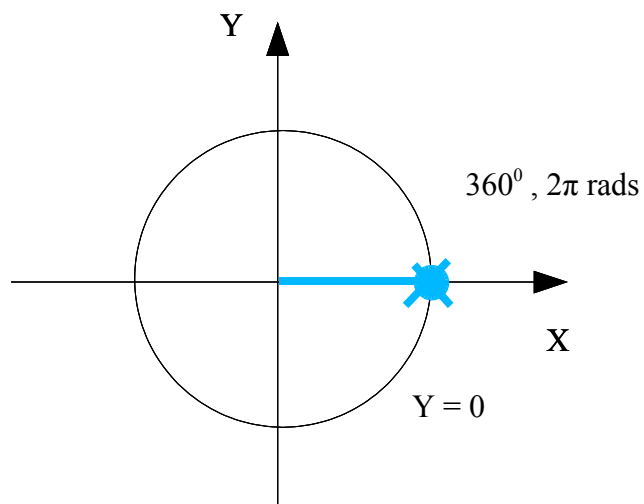
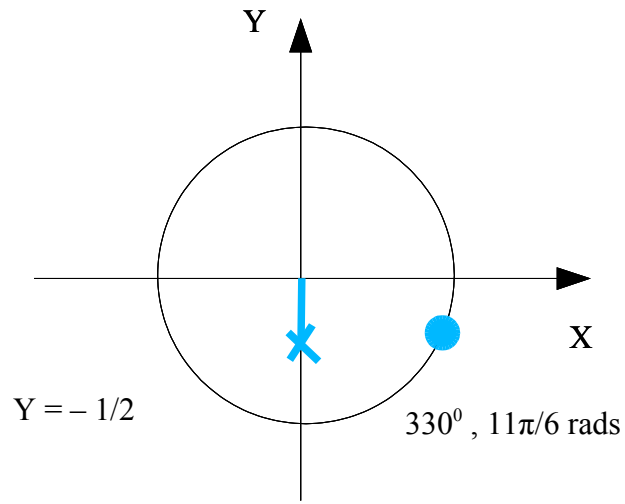






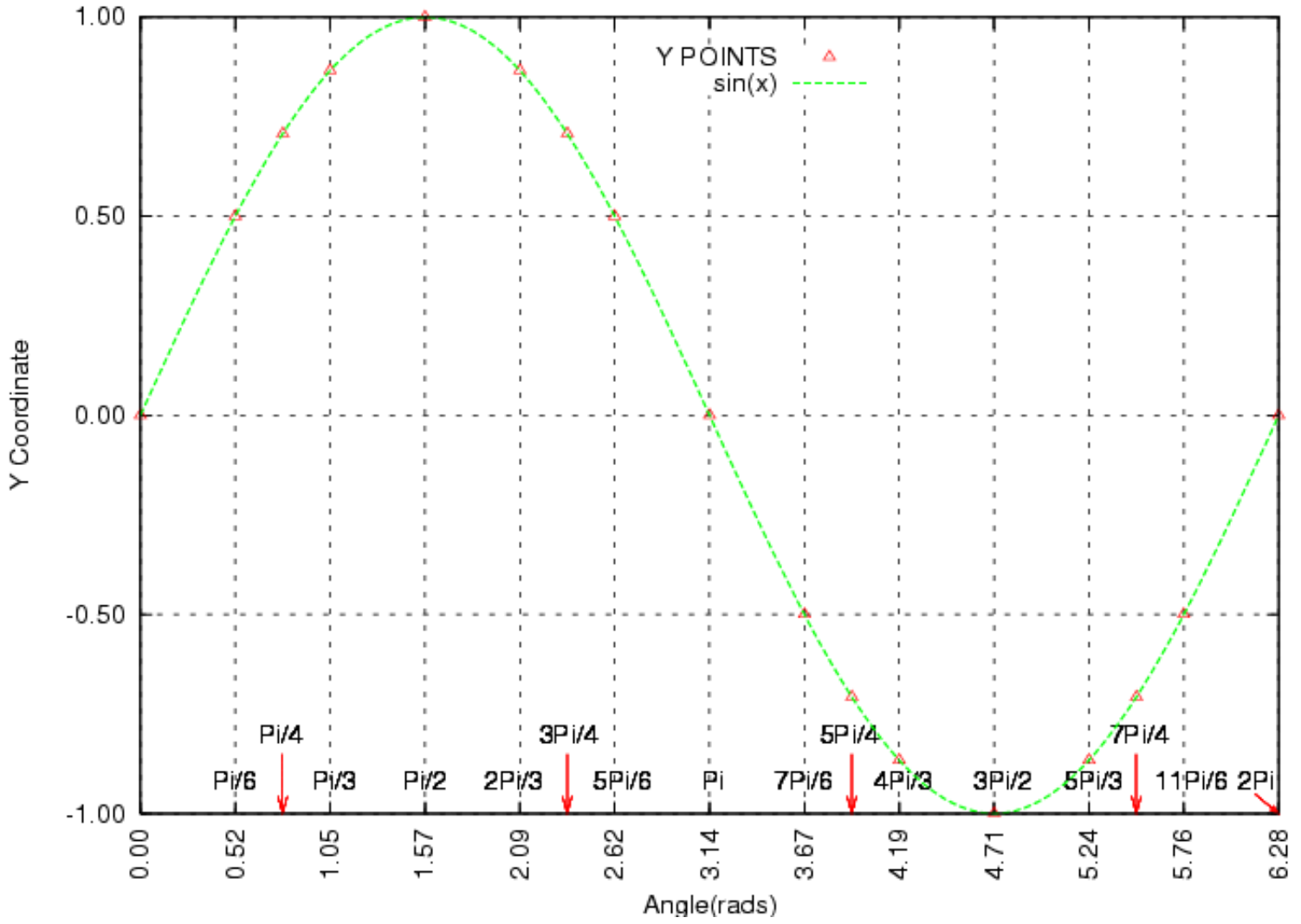




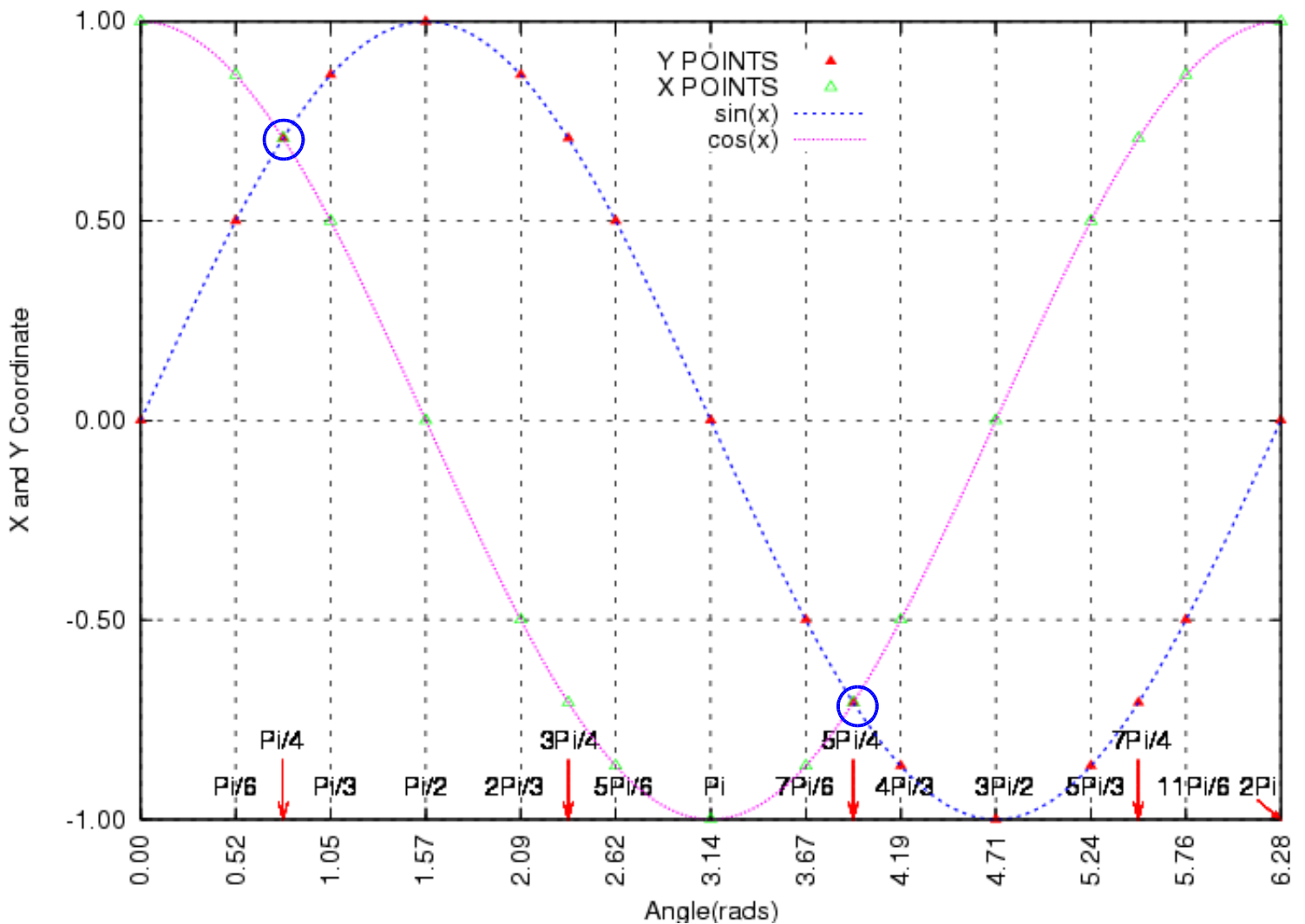


We've completed one revolution, the next page gives a table of these Y values followed by a graph of these Y values versus the angle measured.

<u>Angle degrees</u>	<u>Angle rads</u>	<u>Y</u>
0	0	0
30	$\pi/6$	$1/2$
45	$\pi/4$	$(\sqrt{2})/2$
60	$\pi/3$	$(\sqrt{3})/2$
90	$\pi/2$	1
120	$2\pi/3$	$(\sqrt{3})/2$
135	$3\pi/4$	$(\sqrt{2})/2$
150	$5\pi/6$	$1/2$
180	π	0
210	$7\pi/6$	$1/2$
225	$5\pi/4$	$-(\sqrt{2})/2$
240	$4\pi/3$	$-(\sqrt{3})/2$
270	$3\pi/2$	-1
300	$5\pi/3$	$-(\sqrt{3})/2$
315	$7\pi/4$	$-(\sqrt{2})/2$
330	$11\pi/6$	$-1/2$
360	2π	0



Placing both plots on one graph shows how both the X and Y coordinates change as the angle changes from 0 to 2π through the common angles used in the preceding tables. As it turns out, one only needs to know the values for cos and sin for the common angles in the first quadrant; a sign change will give the value for the corresponding angles in the other three quadrants (more on this later).



Observation 1: As the magnitude of one function increases the magnitude of the other function decreases. as $|\cos(\alpha)| \uparrow$ $|\sin(\alpha)| \downarrow$
and as $|\sin(\alpha)| \uparrow$ $|\cos(\alpha)| \downarrow$

Observation 2: The blue circles show where $\cos(\alpha) = \sin(\alpha)$ $\alpha = \pi/4$ and $5\pi/4$ (45° and 225°)
(at $3\pi/4$ and $7\pi/4$ (135° and 315°)) $\cos(\alpha)$ and $\sin(\alpha)$ are equal but opposite in sign

Observation 3: When $\cos(\alpha) = 0$, $\sin(\alpha) = 1$ or -1 and when $\sin(\alpha) = 0$, $\cos(\alpha) = 1$ or -1

ANGLE α in degrees or rads

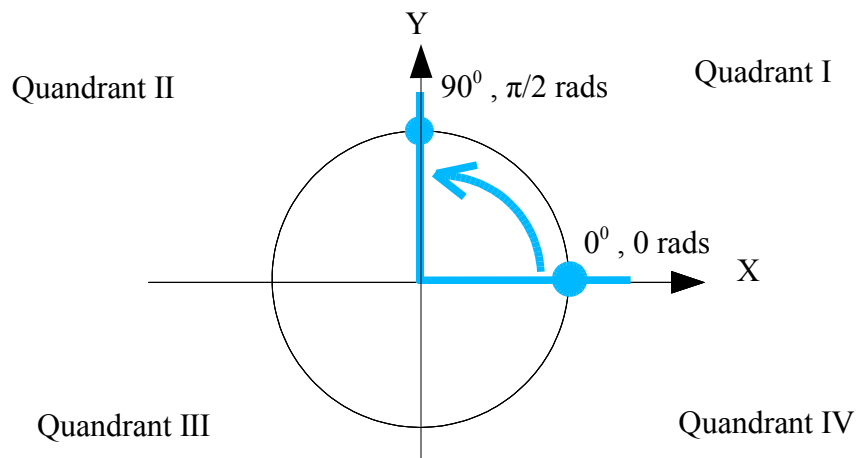
$X = \cos(\alpha)$

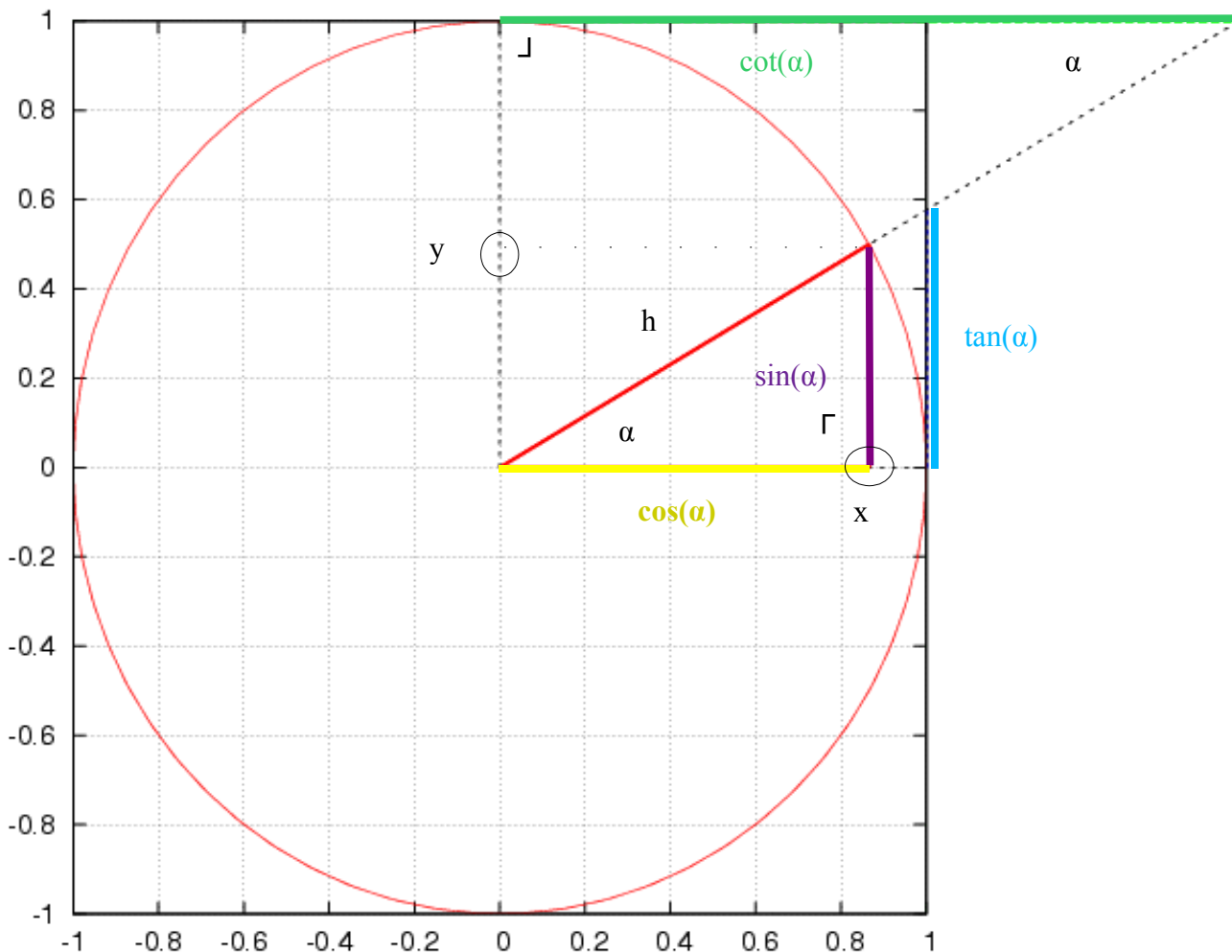
$Y = \sin(\alpha)$

degrees	rads	X	Y
0	0	1	0
30	$\pi/6$	$(\sqrt{3})/2$	$1/2$
45	$\pi/4$	$(\sqrt{2})/2$	$(\sqrt{2})/2$
60	$\pi/3$	$1/2$	$(\sqrt{3})/2$
90	$\pi/2$	0	1
120	$2\pi/3$	$-1/2$	$(\sqrt{3})/2$
135	$3\pi/4$	$-(\sqrt{2})/2$	$(\sqrt{2})/2$
150	$5\pi/6$	$-(\sqrt{3})/2$	$1/2$
180	π	-1	0
210	$7\pi/6$	$-(\sqrt{3})/2$	$1/2$
225	$5\pi/4$	$-(\sqrt{2})/2$	$-(\sqrt{2})/2$
240	$4\pi/3$	$-1/2$	$-(\sqrt{3})/2$
270	$3\pi/2$	0	-1
300	$5\pi/3$	$1/2$	$-(\sqrt{3})/2$
315	$7\pi/4$	$(\sqrt{2})/2$	$-(\sqrt{2})/2$
330	$11\pi/6$	$(\sqrt{3})/2$	$-1/2$
360	2π	1	0

The values in the blue box are in the *first quadrant* of the unit circle, and repeat with a possible sign change moving through quadrants II, III, and IV. Knowing the first quadrant values is sufficient to knowing all other corresponding angles in the remaining three quadrants.

Understanding this relationship between these common angles is the key to understanding and becoming proficient in trigonometry.





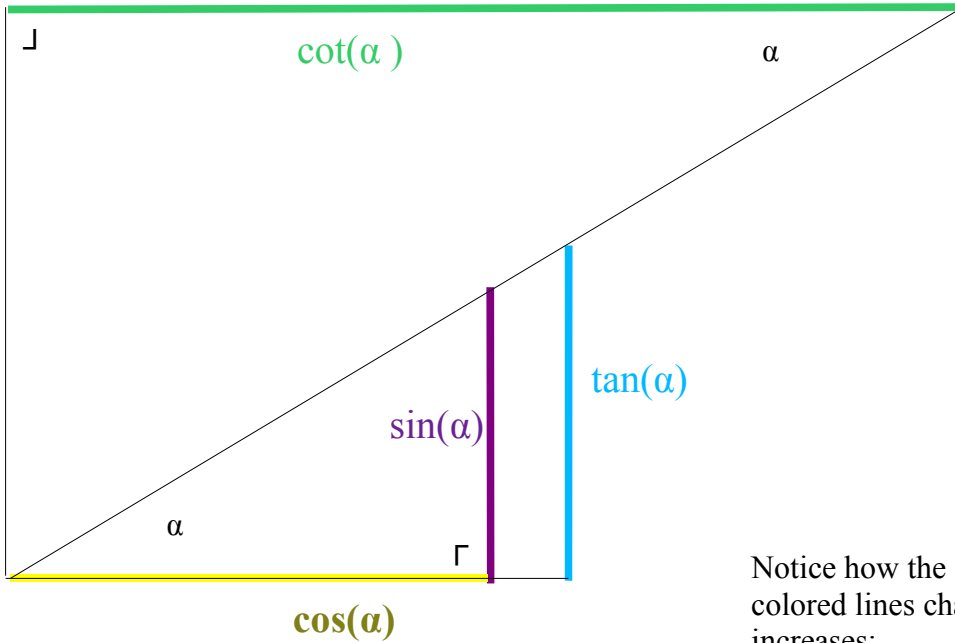
The figure above illustrates how the fundamental trigonometric functions relate to the unit circle in Quadrant I. The right triangle inside the circle (red, yellow, magenta sides) includes the angle α , this angle is called the **working angle**. The length of the radius of this circle and therefore the length of the hypotenuse of this triangle (red line) is 1. Every triangle similar to this triangle will have equal ratios of sides. In particular these ratios will equal $\cos(\alpha)$ and $\sin(\alpha)$ and $\tan(\alpha)$.

$$\begin{aligned} \text{The ratios } \quad \mathbf{x / h} &= \cos(\alpha) / 1 &= \mathbf{\cos(\alpha)} \\ \mathbf{y / h} &= \sin(\alpha) / 1 &= \mathbf{\sin(\alpha)} \\ \mathbf{y / x} &= \sin(\alpha) / \cos(\alpha) = \tan(\alpha) / 1 &= \mathbf{\tan(\alpha)} \end{aligned}$$

(similar triangles with cyan leg and X = 1)

are the fundamental ratios used to solve right triangles.

Notice the larger triangle with the green side. The length of this side is $1 / \tan(\alpha)$.
 (similar triangles: $\cot(\alpha) / 1 = 1 / \tan(\alpha) \rightarrow \cot(\alpha) = 1 / \tan(\alpha)$) “cot” == “cotangent”



Notice how the lengths of the colored lines change as α increases:

$$\cos(\alpha) \downarrow$$

$$\sin(\alpha) \uparrow$$

$$\tan(\alpha) \uparrow$$

$$\cot(\alpha) \downarrow$$

as α approaches $\pi/2$ (90°)

$\cos(\alpha)$ decreases to 0

$\sin(\alpha)$ increases to 1

$\cot(\alpha)$ decreases to 0

$\tan(\alpha)$ increases without bound

and at $\alpha=0$ $\tan(\alpha)$ is undefined.

